Obviously much of our previous discussion of the symmetries carries over straightforwardly to the case of the n-sphere. The condition (6.102) is invariant under rotations defined by

$$X'^a = M_{ab}\, X^b\,, \tag{6.103}$$

where M_{ab} is an $O(n+1)$ matrix satisfying

$$M_{ab}\, M_{ac} = \delta_{bc}\,. \tag{6.104}$$

Infinitesimally we can again write $M_{ab} = \delta_{ab} + A_{ab}$, where the infinitesimal matrix A_{ab} is antisymmetric. This matrix has $\frac{1}{2}n(n+1)$ independent components, so we conclude that the dimension of the group $O(n+1)$ is

$$\dim(O(n+1)) = \tfrac{1}{2}n\,(n+1)\,. \tag{6.105}$$

By the dimension of the group, we mean the number of continuous parameters needed to specify a group element; we saw for $O(3)$ that the answer was 3. As in the case of $O(3)$, the group elements divide into those that have determinant $+1$, and those that have determinant -1. The former correspond to pure rotations in $\mathbb{R}^{n+1}$, while the latter correspond to rotations together with a reflection. Since the identity element obviously has determinant $+1$ it follows that all the infinitesimal transformations must be contained in $SO(n+1)$ too.

It would be quite complicated to generalise the spherical polar coordinates that we used on S^2 to the case of S^n, but in fact for many purposes we can perfectly well just use the Cartesian coordinates X^a on $\mathbb{R}^{n+1}$, together with the constraint (6.102). For example, we can write the infinitesimal $SO(n+1)$ transformations as $\delta X^a = \xi^a$, where $\xi^a = A_{ab}\, X^b$. Thus we are led to the Killing vectors K_{ab}, defined by

$$K_{ab} \equiv X^a\, \frac{\partial}{\partial X^b} - X^b\, \frac{\partial}{\partial X^a}\,, \tag{6.106}$$

where the ab indices here are labels, indicating which Killing vector we mean. By construction we have $\frac{1}{2}n(n+1)$ Killing vectors, since $K_{ab} = -K_{ba}$. This is the correct number for the $SO(n+1)$ symmetry of the n-sphere. If we specialise to the 2-sphere, it is easy to verify that the three Killing vectors K_{12}, K_{13} and K_{23} defined by (6.106) in this case are just the same, after the change to spherical polar coordinates, as the Killing vectors (6.32) that we derived previously.

Notice that the Killing vectors (6.106) are nothing but the angular momentum operators in $(n+1)$-dimensional Euclidean space. In 3 dimensions we would more commonly use the totally-antisymmetric epsilon tensor ϵ_{abc} to re-express the angular momentum operators in terms of a vector index:

$$L_a = \tfrac{1}{2}\epsilon_{abc}\, K_{bc} = \epsilon_{abc}\, X^b\, \frac{\partial}{\partial X^c}\,. \tag{6.107}$$

Observe, though, that it is a very special feature of 3 dimensions that one can replace an antisymmetric 2-index quantity like K_{ab} by a vector. In higher dimensions, where the corresponding totally-antisymmetric epsilon tensor has more indices, one cannot turn a 2-index antisymmetric tensor into a tensor with fewer indices. In fact this serves to emphasise that in a general dimension one should think of rotations as occurring *in planes*, rather than *about axes*. It is a coincidence of 3 dimensions that a rotation in the (x, y) plane can also be thought of as a rotation about the z axis.

6.5.2 Spherical Harmonics

When one first meets the spherical harmonics on the 2-sphere, it is generally in the context of performing a separation of variables in Laplace's equation or the wave equation, when using spherical polar coordinates. In fact we just re-derived the expression for this Laplacian in the previous section, in (6.101). After a standard separation of variables in which a function $\psi(r,\theta,\phi)$ is written as

$$\psi(r,\theta,\phi) = R(r)\, Y(\theta,\phi)\,, \tag{6.108}$$

Laplace's equation $\nabla^2\psi = 0$ becomes

$$\frac{1}{R}\,\frac{d}{dr}\Big(r^2\,\frac{dR}{dr}\Big) + \frac{1}{Y}\,\nabla^2_{S^2}\, Y = 0\,, \tag{6.109}$$

where $\nabla^2_{S^2}$ is the operator appearing in the large square brackets in (6.101), namely

$$\nabla^2_{S^2} = \frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\Big(\sin\theta\,\frac{\partial}{\partial\theta}\Big) + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\phi^2}\,. \tag{6.110}$$

In fact this operator is precisely the Laplacian for the unit 2-sphere, as may easily be checked by using our general formula (6.97), with the metric $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. Introducing a separation constant λ in the usual way, one is led from (6.109) to consider the equation

$$-\nabla^2_{S^2}\, Y(\theta,\phi) = \lambda\, Y(\theta,\phi)\,. \tag{6.111}$$

This is the equation that determines the spherical harmonics.

A standard way to solve for the spherical harmonics is to write out the S^2 Laplacian $\nabla^2_{S^2}$ explicitly using (6.110), and perform a further separation of variables by writing $Y(\theta,\phi) =$

$P(\theta)\,\Phi(\phi)$. This introduces another separation constant m^2, an one is left to solve the equations

$$\begin{aligned}\sin\theta\,\frac{d}{d\theta}\Big(\sin\theta\,\frac{dP}{d\theta}\Big)+(\lambda\,\sin^2\theta-m^2)\,P &= 0\,,\\ \frac{d^2\Phi}{d\phi^2}+m^2\,\Phi &= 0\,.\end{aligned}\tag{6.112}$$

The latter has solutions of the form $e^{{\rm i}\,m\,\phi}$, and to get the proper periodicity under complete rotations $\phi\longrightarrow\phi+2\pi$ on the sphere, we deduce that m must be an integer. After letting $x=\cos\theta$ the first equation becomes the generalised Legendre equation,

$$\frac{d}{dx}\Big((1-x^2)\,\frac{dP}{dx}\Big)+\Big(\lambda-\frac{m^2}{1-x^2}\Big)\,P=0\,.\tag{6.113}$$

After a considerable labour, involving, for example, a careful study of the solutions for this equation obtained as a series expansion (discussed at length in Part I of the course), one concludes that for the functions $P(\theta)$ to be regular at $\theta=0$ and π (the north and south poles of the sphere), the separation constant λ must be of the form $\lambda=\ell\,(\ell+1)$, where ℓ is an integer, and $-\ell\le m\le\ell$. Thus after a rather involved chain of argument, one arrives at the spherical harmics $Y_{\ell m}(\theta,\phi)$ being the complet set of regular eigenfunctions of the Laplacian $\nabla^2_{S^2}$ on S^2, with

$$-\nabla^2_{S^2}\,Y_{\ell m}=\ell\,(\ell+1)\,Y_{\ell m}\,.\tag{6.114}$$

Of course one has the feature that since m does not appear in the expression for the eigenvalues, there is a $(2\ell+1)$-fold degeneracy for the spherical harmonics with a given value of ℓ, since m can take any of the integer values between $-\ell$ and $+\ell$.

This traditional approach to constructing the spherical harmonics is a rather calculational one, which provides very little group-theoretic insight into what is going on. We are in fact now in a position to give a much simpler, and more elegant, construction of the spherical harmonics, which provides us with a rather clear picture of them as representations of the symmetry group $SO(3)$ of the 2-sphere. In fact it is just as easy to construct the spherical harmonics on all the spheres S^n, for arbitrary dimension n, so there is that advantage too.

We have described the unit n-sphere as the surface $X^a\,X^a=1$ in $\mathbb{R}^{n+1}$. Let us write the metric on the unit n-sphere as $d\Omega^2$. It is evident that this is related to the Cartesian metric ds^2 on $\mathbb{R}^{n+1}$ by

$$ds^2=dr^2+r^2\,d\Omega^2\,,\tag{6.115}$$

where $X^a\,X^a=r^2$. This is clear, if you think about how we would measure distances in $\mathbb{R}^{n+1}$ if it were written in "hyperspherical polar coordinates," r and y^α, where y^α represent the set of angular that one would use to parameterise points on the unit n-sphere. The square of the distance between two infinitesimally separated points in $\mathbb{R}^{n+1}$ is therefore the sum of the square of the radial-coordinate separation dr, and the square of the distance in the surface of the sphere that separates the two points. Since $d\Omega^2$ is the metric on the unit sphere, the distance on the sphere of radius r, where the two points are located, will be scaled by the factor r. It is easy to see that (6.115) reduces to familiar cases if we consider $\mathbb{R}^2$ and $\mathbb{R}^3$, since the metrics on the unit 1-sphere and 2-sphere are just

$$\begin{aligned} \text{1-sphere}: \qquad & d\Omega^2 = d\theta^2\,, \\ \text{2-sphere}: \qquad & d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2\,. \end{aligned} \tag{6.116}$$

Luckily we don't ever need to define the angular coordinates on S^n explicitly, in order to solve for the spherical harmonics. We can just let them be called y^α, with $1 \le \alpha \le n$, but we don't need to define how they are related to the Cartesian coordinates X^a in $\mathbb{R}^{n+1}$. (One can usefully have in mind, though, the picture that they will be defined very analogously to the way spherical polar coordinates are related to the (X, Y, Z) coordinates on $\mathbb{R}^3$.) The metric on the unit n-sphere can then be written as

$$d\Omega^2 = h_{\alpha\beta}\, dy^\alpha\, dy^\beta\,. \tag{6.117}$$

The full set of $(n+1)$ hyperspherical coordinates on $\mathbb{R}^{n+1}$ will be (r, y^α). Let us call these hyperspherical coordinates x^i, with i running from 0 to n:

$$x^0 \equiv r\,, \qquad x^\alpha \equiv y^\alpha\,. \tag{6.118}$$

Now, using (6.117), the metric (6.115) on $\mathbb{R}^{n+1}$ is

$$ds^2 = dr^2 + r^2\, h_{\alpha\beta}\, dy^\alpha\, dy^\beta\,. \tag{6.119}$$

Clearly therefore the determinant g of this metric is given by

$$g = r^n\, h\,, \tag{6.120}$$

where h is the determinant of the metric $h_{\alpha\beta}$ on the unit n-sphere. Plugging into our general expression (6.97) for the Laplacian, we therefore find that in these hyperspherical polar coordinates, the Laplacian on $\mathbb{R}^{n+1}$ is given by

$$\nabla^2_{R^{n+1}} = \frac{1}{r^n}\,\frac{\partial}{\partial r}\Big(r^n\,\frac{\partial}{\partial r}\Big) + \frac{1}{r^2}\,\nabla^2_{S^n}\,, \tag{6.121}$$

where

$$\nabla^2_{S^n} \equiv \frac{1}{\sqrt{h}}\,\frac{\partial}{\partial y^\alpha}\Big(\sqrt{h}\, h^{\alpha\beta}\,\frac{\partial}{\partial y^\beta}\Big) \tag{6.122}$$

is the Laplacian on the unit n-sphere. (The special cases for $n = 1$ and $n = 2$ appear in our examples (6.98) and (6.101) that we looked at previously.)

Having obtained this relation between the Laplacians on $\mathbb{R}^{n+1}$ and S^n, the problem of constructing the spherical harmonics is almost solved. First, we introduce the following functions Ψ on $\mathbb{R}^{n+1}$:

$$\Psi(X) = T_{a_1 a_2 \cdots a_\ell}\, X^{a_1}\, X^{a_2} \cdots X^{a_\ell}\,, \tag{6.123}$$

where $T_{a_1 a_2 \cdots a_\ell}$ is an ℓ-index constant tensor in $\mathbb{R}^{n+1}$ which is completely arbitrary except for satisfying the following two conditions:

(1) $T_{a_1 a_2 \cdots a_\ell}$ is *totally symmetric* in all its indices.

(2) The tensor T is totally traceless, in the sense that the contraction of any pair of indices on $T_{a_1 a_2 \cdots a_\ell}$ gives zero:

$$\delta_{a_1 a_1}\, T_{a_1 a_2 \cdots a_\ell} = 0\,, \quad \text{etc.}\,. \tag{6.124}$$

Clearly condition (1) is simply making sure that we eliminate all the "redundant baggage" in $T_{a_1 a_2 \cdots a_\ell}$. Since it appears in (6.123) contracted onto the totally symmetrical product $X^{a_1}\, X^{a_2} \cdots X^{a_\ell}$, it is obvious that any part of $T_{a_1 a_2 \cdots a_\ell}$ that was not totally symmetrical in the indices would give no contribution anyway.

Condition (2) serves a different purpose. It implies that if we act with the $\mathbb{R}^{n+1}$ Laplacian $\nabla^2_{R^{n+1}}$ on Ψ, we shall get zero:

$$\nabla^2_{R^{n+1}}\, \Psi = 0\,. \tag{6.125}$$

This is because from the definition of Ψ in (6.123), we shall clearly have

$$\begin{aligned} \frac{\partial \Psi}{\partial X_a} &= T_{a a_2 \cdots a_\ell}\, X^{a_2} \cdots X^{a_\ell} + T_{a_1 a \cdots a_\ell}\, X^{a_1}\, X^{a_3} \cdots X^{a_\ell} + \cdots T_{a_1 a_2 \cdots a_\ell}\, X^{a_1}\, X^{a_2} \cdots X^{a_{\ell-1}} \\ &= \ell\, T_{a a_2 \cdots a_\ell}\, X^{a_2} \cdots X^{a_\ell}\,, \end{aligned} \tag{6.126}$$

(all the ℓ terms are equal, because of the total symmetry). Acting with another derivative, we therefore get

$$\frac{\partial^2 \Psi}{\partial X^a\, \partial X^b} = \ell\, (\ell - 1)\, T_{a b a_3 \cdots a_\ell}\, X^{a_3} \cdots X^{a_\ell}\,. \tag{6.127}$$

(This time, we have immediately used the symmetry of T to collect the $(\ell - 1)$ terms that appear from the second differentiation together. Now we see that the $\mathbb{R}^{n+1}$ Laplacian acting on Ψ gives zero:

$$\nabla^2_{R^{n+1}}\, \Psi = \frac{\partial^2 \Psi}{\partial X^a \partial X^a} = \ell\, (\ell - 1)\, \delta_{ab}\, T_{a b a_3 \cdots a_\ell}\, X^{a_3} \cdots X^{a_\ell} = 0\,, \tag{6.128}$$

by virtue of condition (2) above.

Now, it only remains to make the following observation. Since the function Ψ defined in (6.123) involes a product of ℓ Cartesian coordinates X^a, it is evident that it must be expressible as

$$\Psi(X) = r^\ell\, \psi(y)\,, \tag{6.129}$$

where y represents the angular coordinates y^α on the unit n-sphere, and $\psi(y)$ is *independent of* r. Again, it is helpful to have in mind the $\mathbb{R}^3$ example here, where we have

$$X = r\,\sin\theta\,\cos\phi\,,\qquad Y = r\,\sin\theta\,\sin\phi\,,\qquad Z = r\,\cos\theta\,. \tag{6.130}$$

Finally, since we have established that the $\mathbb{R}^{n+1}$ Laplacian annihilates Ψ we simply have to substitute it into (6.121) to deduce that

$$\frac{1}{r^n}\,\frac{d}{dr}\Big(r^n\,\frac{dr^\ell}{dr}\Big)\,\psi + \frac{1}{r^2}\,r^\ell\,\nabla^2_{S^n}\,\psi = 0\,. \tag{6.131}$$

Hence we arrive at the conclusion that ψ is an eigenfunction of the Laplacian on the unit n-sphere, satisfying

$$-\nabla^2_{S^n}\,\psi = \ell\,(\ell+n-1)\,\psi\,. \tag{6.132}$$

Notice that is we take $n=2$, corresponding to the 2-sphere, we reproduce the familiar eigenvalues $\ell\,(\ell+1)$.

Two issues remain to be discussed here. The first is that we have certainly produced *some* eigenfunctions on the n-sphere by this method, but have we obtained them all? The answer is yes, and it can be seen as follows. Clearly, any regular function on the unit n-sphere can be smoothly extended out as a regular function on $\mathbb{R}^{n+1}$. Conversely, if we consider the set of all regular functions on $\mathbb{R}^{n+1}$, they will project down so as to provide us with all possible regular functions on S^n. Now, the regular functions $f(X)$ on $\mathbb{R}^{n+1}$ can certainly be expanded in a Taylor series, which will give a sum of terms of the form (6.123), summed over all $\ell \ge 0$ (without yet imposing the tracelessness of condition (2) above):

$$f(X) = \sum_{\ell=0}^{\infty} f_\ell(X)\,, \tag{6.133}$$

where

$$f_\ell(X) \equiv T_{a_1 a_2\cdots a_\ell}\,X^{a_1}\,X^{a_2}\cdots X^{a_\ell}\,, \tag{6.134}$$

But the imposition of tracelessness on $T_{a_1 a_2\cdots a_\ell}$ is just a matter of organising the terms in the sum, since a pure trace contribution in the term $f_\ell(X)$ would correspond to r^2 times a term of the form $f_{\ell-2}(X)$. By the time we restricted to the unit n-sphere, by setting $r=1$, this from f_ℓ term would therefore just be repeating what had already been constructed in $f_{\ell-2}$. So from the viewpoint of constructing regular functions on the n-sphere, the imposition of tracelessness on the tensors $T_{a_1 a_2 \cdots a_\ell}$ is just a matter of avoiding double-counting. Thus we can be sure that our construction of scalar eigenfunctions of the Laplacian on S^n has given *all* all the eigenfunctions. The functions ψ, defined by (6.123) and (6.129), then, give the complete set of *spherical harmonics* on S^n.

The second issue that we must still address concerns the degneracies of the eigenvalues, or, equivalently, the *multiplicities* of the eigenfunctions ψ for a given value of the integer ℓ. This is easily worked out, since it is just a matter of counting how many independent components the constant tensor $T_{a_1 a_2 \cdots a_\ell}$ has, bearing in mind the two conditions of symmetry and tracelessness that we imposed. It is easy to see that a totally-symmetric tensor with ℓ indices that each run over $(n+1)$ values has

$$\frac{(n+1)(n+2)\cdots(n+\ell)}{\ell!} \tag{6.135}$$

independents components. When we impose the traceless condition on such a tensor, we therefore impose a number of conditions equal to the number of independent components in a similar tensor that has only $(\ell-2)$ indices. Thus the number of independent components in our tensor $T_{a_1 a_2 \cdots a_\ell}$ that is totally symmetric and traceless is

$$\begin{aligned} d_\ell &= \frac{(n+1)(n+2)\cdots(n+\ell)}{\ell!} - \frac{(n+1)(n+2)\cdots(n+\ell-2)}{(\ell-2)!}\,, \\ &= \frac{(n+1)(n+2)\cdots(n+\ell-2)}{\ell!}\Big((n+\ell-1)({\scriptstyle (n)}+\ell) - \ell\,(\ell-1)\Big)\,, \\ &= \frac{n\,(n+1)(n+2)\cdots(n+\ell-2)(2\ell+n-1)}{\ell!}\,, \end{aligned} \tag{6.136}$$

which can be written as

$$d_\ell = \frac{(2\ell+n-1)\,(n+\ell-2)!}{\ell!\,(n-1)!}\,. \tag{6.137}$$

This gives us the multiplicity of the eigenfunctions ψ with the specific eigenvalue

$$\lambda_\ell = \ell\,(\ell+n-1) \tag{6.138}$$

that we found above. Notice that if we specialise to the case of the 2-sphere, equation (6.137) reduces to

$$\text{2-sphere:} \qquad d_\ell = 2\ell + 1\,, \tag{6.139}$$

as we know it should.

6.5.3 Irreducible Representations of $SO(N)$

The construction of the eigenfunctions that we have obtained here, and the results for the multiplicities of the eigenvalues, have a deeper significance than might at first be apparent. What we have actually been doing here is constructing *irreducible representations* of the the symmetry groups $SO(n+1)$ of the n-spheres. To be a bit more precise, the sets of tensors $T_{a_1 a_2 \cdots a_\ell}$ that we have been using are themselves irreducible representations of $SO(n+1)$. More generally, one can consider many different classes of constant tensor $H_{a_1 a_2 \cdots a_p}$ in $\mathbb{R}^{n+1}$, and associate them with irreducible representations.

To make life a little simpler, let us talk about $SO(N)$ rather than $SO(n+1)$. If we begin with the tensor $H_{a_1 a_2 \cdots a_p}$ in $\mathbb{R}^N$, and make no symmetry or tracelessness requirement at all on it, then the number of independent components for such a tensor will simply be N^p, since each index can range over N values. This set of tensors with N^p components is a representation of $SO(N)$, but it is not irreducible; we can divide it into smaller self-contained subsets of components. The rules for how such subdivisions can be made are very simple. We can do anything as long as it respects $SO(N)$ covariance. What this means is that we have to treat the indices in a totally "democratic" way, and we cannot single out any one index value, or subset of index values, for special treatment.

Let us take a concrete example. Suppose we take a 2-index tensor H_{ab} in $\mathbb{R}^N$, which has N^2 independent components. Is this reducible, or is it already as irreducible as can be? First, the sort of things we *cannot* do is to pick an index value, say $a=1$, and treat that as special. We cannot divide H_{ab} into $H_{\alpha\beta}$, $H_{1\alpha}$, $H_{\alpha 1}$ and H_{11}, where $2 \le \alpha \le N$, and claim that we are decomposing H_{ab} into representations of $SO(N)$; clearly what we are doing here is not covariant from an $SO(N)$ point of view. What we *can* do, however, is to write H_{ab} as the sum of its symmetric and antisymmetric parts:

$$H_{ab} = S_{ab} + A_{ab}\,, \tag{6.140}$$

where

$$S_{ab} \equiv \tfrac{1}{2}(H_{ab} + H_{ba})\,, \qquad A_{ab} \equiv \tfrac{1}{2}(H_{ab} - H_{ba})\,. \tag{6.141}$$

Now, we can count the number of independent components in S_{ab}, namely $\frac{1}{2}N(N+1)$, and the number of independent components in A_{ab}, namely $\frac{1}{2}N(N-1)$. Of course the sum of these two gives us back the original number of components for the unrestricted tensor H_{ab}:

$$\tfrac{1}{2}(N(N+1) + \tfrac{1}{2}N(N-1) = N^2\,. \tag{6.142}$$

Clearly the decomposition in (6.140) is completely covariant with respect to $SO(N)$, since it is a tensorial equation, so it is a perfectly allowable subdivision for us to make.

Have we finished? Not quite, because there is one more thing we can do that respects the covariance, and that is to extract the trace from the symmetric tensor S_{ab}. Thus we can write

$$S_{ab} = \widetilde{S}_{ab} + \frac{1}{N}\, S\, \delta_{ab}\,, \tag{6.143}$$

where S is the trace of S_{ab}, namely

$$S \equiv \delta_{ab}\, S_{ab}\, .\,, \tag{6.144}$$

and so by construction $\widetilde{S}_{ab}$ is traceless,

$$\delta_{ab}\, \widetilde{S}_{ab} = 0\,. \tag{6.145}$$

Clearly (6.143) and (6.144) are both perfectly $SO(N)$-covariant equations; they transform covariantly under $SO(N)$ rotations. (We are really back to "kindergarten" Cartesian tensors here!)

With this extraction of the trace, we have reached the end of the road for the decomposition of the original 2-index tensor H_{ab}. In other words, we have found that it splits into three irreducible representations of $SO(N)$, with dimensions

$$\dim(A_{ab}) = \tfrac{1}{2}N(N-1)\,, \qquad \dim(\widetilde{S}_{ab}) = \tfrac{1}{2}(N-1)(N+2)\,, \qquad \dim(S) = 1\,, \tag{6.146}$$

These are the dimensions of the 2-index antisymmetric representation, the 2-index symmetric traceless representation, and the singlet of $SO(N)$ respectively.

The original H_{ab} representation is really to be thought of as the product of two 1-index representations. The 1-index, or *vector representation* of $SO(N)$ corresponds, as its name implies, to taking an arbitrary constant vector H_a in $\mathbb{R}^n$. It is clear that we cannot subdivide this representation any further by means of any allowable covariant rules, and so it is an N-dimensional irreducible representation.

We have just met four different irreducible representations of $SO(N)$, and we have seen that the following multiplication rule applies:

$$\underline{N} \times \underline{N} = \underline{\tfrac{1}{2}N(N-1)} + \underline{\tfrac{1}{2}(N-1)(N+2)} + \underline{1}\,. \tag{6.147}$$

What this is saying is that the product of the vector representation of $SO(N)$ with itself gives the three irreducible representations whose dimensions are listed on the right-hand side. For example, in $SO(3)$ we have

$$\underline{3} \times \underline{3} = \underline{3} + \underline{5} + \underline{1}\,. \tag{6.148}$$

Note that we use the underlining notation to indicate that we are talking about group representation here.[27]

One can continue the process of examining $SO(N)$ tensors with more and more indices, in each case making a covariant decomposition into the largest possible number of irreducible pieces, and thereby one builds up the complete set of irreducible representations of $SO(N)$. It gets a little trickier than the examples we have looked at so far, once the tensor has several indices. For example, consider a 3-index tensor H_{abc}. This certainly contains a totally-symmetric piece, and a totally antisymmetric piece, but it also has more. This can easily be seen by noting that sum of the independent components $\frac{1}{6}N(N+1)(N+2)$ of a symmetric 3-index tensor and the independent components $\frac{1}{6}N(N-1)(N-2)$ of an antisymmetric 3-index tensor does not add up to the N^3 components of an arbitrary 3-index tensor. There is nothing deep or mysterious about this, of course, and it is really just an exercise in symmetries and combinatorics to work out what the "extra" pieces are. Of course one also needs to extract all trace terms where appropriate, and count those as separate irreducible pieces. A very hand diagrammatic method, known as *Young Tableaux*, has been developed for doing all this. However, it takes us beyond the scope of this introductory discussion, so we shall leave it at that.

For our present purposes we don't need anything terribly exotic, because we saw that in the contruction of the spherical harmonics it was the totally symmetric and traceless $SO(n+1)$ tensors $T_{a_1 a_2 \cdots a_\ell}$ that were relevant. What we have now learned from the above discussion is that these tensors are actually giving us irreducible representations of $SO(n+1)$, and we have already worked out their dimensions d_ℓ in (6.137). For the 2-sphere, this became $d_\ell = 2\ell+1$, and so what we are seeing is that the spherical harmonics on S^2 occur in the following irreducible representations of $SO(3)$:

$$d_\ell = 2\ell + 1 = \underline{1}\,, \underline{3}\,, \underline{5}\,, \underline{7}\,, \ldots \qquad (6.149)$$

As the dimension $d_\ell = 2\ell+1$ of the representation gets bigger, so, correspondingly, does the eigenvalue $\lambda_\ell = \ell\,(\ell+1)$.

For the higher-dimensional n-spheres the dimensions of the symmetric traceless irreducible $SO(n+1)$ representations become a bit more interesting. For example, from d_ℓ given in (6.137) we have the following:

$$SO(4): \qquad d_\ell = (\ell+1)^2 = \underline{1}\,, \underline{4}, , \underline{9}\,, \underline{16}\,, \ldots$$

[27] It also serves to show that we are doing profound mathematics here, and that we have not reverted to the kindergarten arithmetic class!